Lecture 9: Symbolic Processing in MATLAB

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The `sym` function can be used to create “symbolic objects” in MATLAB.

If the input argument to `sym` is a string, the result is a symbolic number or variable. If the input argument is a numeric scalar or matrix, the result is a symbolic representation of the given numeric values.

For example, typing `x = sym('x')` creates the symbolic variable with name `x`, and typing `y = sym('y')` creates a symbolic variable named `y`.

Typing `x = sym('x', 'real')` tells MATLAB to assume that `x` is real. Typing `x = sym('x', 'unreal')` tells MATLAB to assume that `x` is not real.

The `syms` function enables you to combine more than one such statement into a single statement.

For example, typing `syms x` is equivalent to typing `x = sym('x')`, and typing `syms x y u v` creates the four symbolic variables `x`, `y`, `u`, and `v`.

Symbolic vs. Numeric Objects

```
>> x = sym('x')
x =
 x
>> class(x)
ans =
 sym
```

```
>> syms y
>> class(y)
ans =
 sym
```

```
>> a = 5
a =
     5
>> class(5)
ans =
 double
```

```
>> b = 't'
b =
 t
>> class(b)
ans =
 char
```

You can use the `sym` function to create *symbolic constants* by using a numerical value for the argument. For example, typing

```
fraction = sym('1/3')

sqroot2 = sym('sqrt(2)')

pi = sym('pi')
```

will create symbolic constants that avoid the floating-point approximations inherent in the values of π, 1/3, and $\sqrt{2}$.

Symbolic Expressions

You can use symbolic variables in expressions and as arguments of functions. You use the operators `+ - * / ^` and the built-in functions just as you use them with numerical calculations. For example, typing

```
>> syms x y
>> s = x + y;
>> r = sqrt(x^2 + y^2);
```

creates the symbolic variables `s` and `r`. The terms `s = x + y` and `r = sqrt(x^2 + y^2)` are examples of symbolic *expressions*.

The vector and matrix notation used in MATLAB also applies to symbolic variables. For example, you can create a symbolic matrix `A` as follows:

```
>> n = 3;
>> syms x;
>> A = x.^((0:n)'*(0:n))
A =
    [ 1, 1, 1, 1]
    [ 1, x, x^2, x^3]
    [ 1, x^2, x^4, x^6]
    [ 1, x^3, x^6, x^9]
```

The `expand` and `simplify` functions.

```
>> syms x y
>> expand((x+y)^2) % applies algebra rules
ans =
     x^2 + 2*x*y + y^2

>> syms x y
>> expand(sin(x+y)) % applies trig identity
ans =
     cos(x)*sin(y) + cos(y)*sin(x)

>> syms x
>> simplify(6*((sin(x))^2+(cos(x))^2))
% applies another trig identity
ans =
     6
```

```
>> syms x
>> E1 = x^2+5;
>> E2 = x^3+2*x^2+5*x+10;
>> S = E1/E2;
>> simplify(S)
ans =
   1/(x + 2)
```

The `factor` function.

```
>> syms x
>> factor(x^2-1)
ans =
     (x - 1)*(x + 1)
```

The function `subs(E,old,new)` substitutes `new` for `old` in the expression `E`, where `old` can be a symbolic variable or expression and `new` can be a symbolic variable, expression, or matrix, or a numeric value or matrix. For example,

```
>> syms x y
>> E = x^2+6*x+7;
>> F = subs(E,x,y)
F =
   y^2 + 6*y + 7

>> G = subs(E,x,y+3)
G =
   6*y + (y + 3)^2 + 25
```

If you want to tell MATLAB that f is a function of the variable t, type `f = sym('f(t)')`. Thereafter, `f` behaves like a function of `t`, and you can manipulate it with the toolbox commands. For example, to create a new function $g(t) = f(t + 2) - f(t)$, the session is

```
>> syms t
>> f = sym('f(t)');
>> g = subs(f,t,t+2)-f
g =
   f(t+2)-f(t)
```

Once a specific function is defined for $f(t)$, the function $g(t)$ will be available.

Use the `subs` and `double` functions to evaluate an expression numerically. Use `subs(E,old,new)` to replace `old` with a numeric value `new` in the expression `E`. The result is of class double. For example,

```
>> syms x
>> E = x^2+6*x+7;
>> G = subs(E,x,2)
G =
   23
>> class(G)
ans =
     double
```

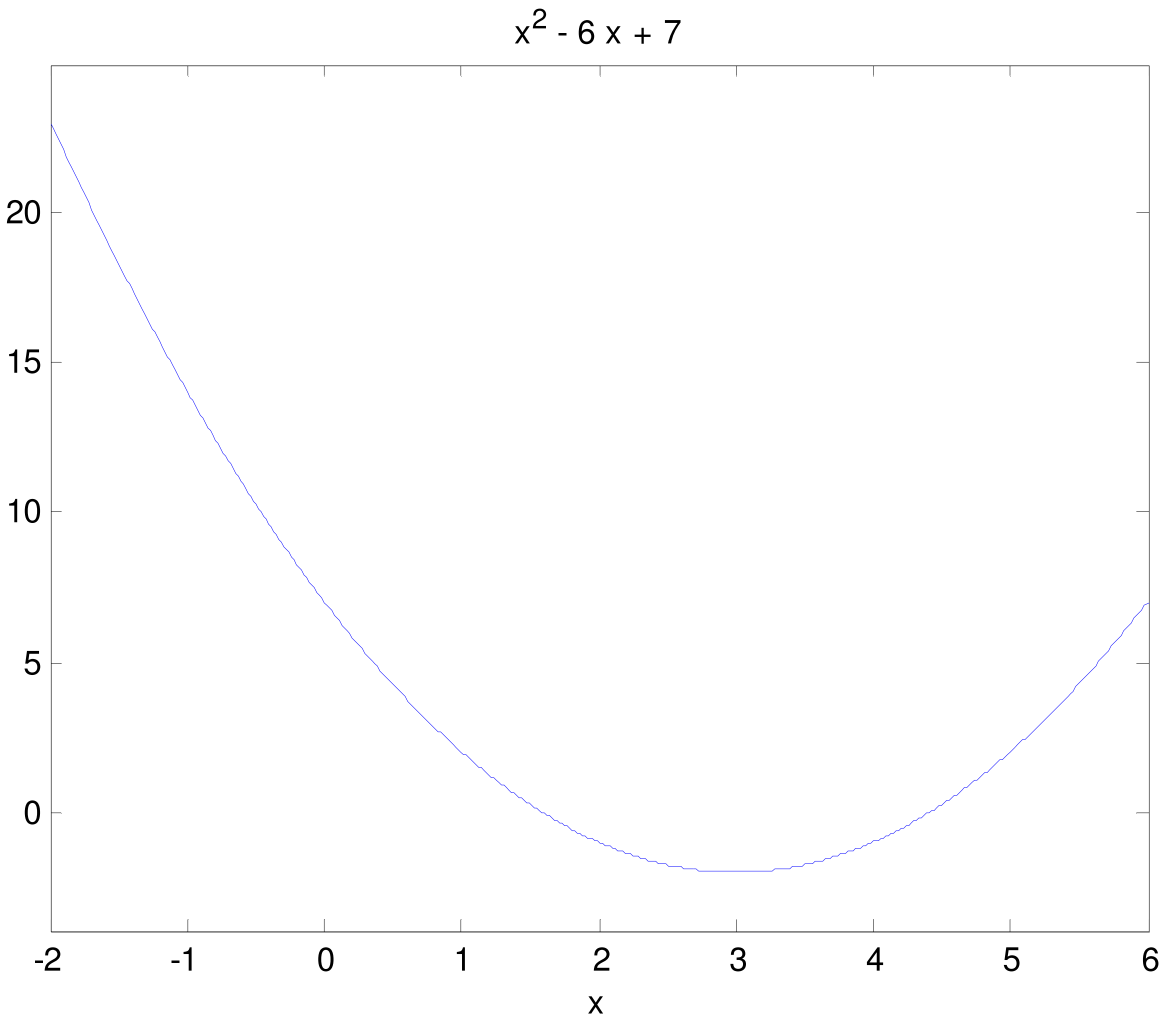
The MATLAB function `ezplot(E)` generates a plot of a symbolic expression `E`, which is a function of one variable. The default range of the independent variable is the interval $[-2\pi, 2\pi]$ unless this interval contains a singularity.

The optional form `ezplot(E,[xmin xmax])` generates a plot over the range from `xmin` to `xmax`.

Example:

```
>> syms x
>> E = x^2 - 6*x + 7;
>> ezplot(E, [-2 6]);
```

Result



Order of Precedence.

MATLAB does not always arrange expressions in a form that we normally would use.

For example, MATLAB might provide an answer in the form `-c+b`, whereas we would normally write `b-c`.

The order of precedence used by MATLAB must be constantly kept in mind to avoid misinterpreting the MATLAB output (see earlier slides).

MATLAB frequently expresses results in the form `1/a*b`, whereas we would normally write `b/a`.

The `solve` function.

There are three ways to use the `solve` function. For example, to solve the equation $x + 5 = 0$, one way is

```
>> eq1 = 'x+5=0';
>> solve(eq1)
ans =
     -5
```

The second way is

```
>> solve('x+5=0')
ans =
     -5
```

The `solve` function (continued).

The third way is

```
>> syms x
>> solve(x+5)
ans =
     -5
```

You can store the result in a named variable as follows:

```
>>syms x
>>x = solve(x+5)
x =
   -5
```

To solve the equation $e^{2x} + 3e^{x} = 54$, the session is

```
>> solve('exp(2*x)+3*exp(x) = 54')
ans =
         log(6)
 log(9) + pi*I
```

```
>> syms x
>> solve(exp(2*x)+3*exp(x)-54)
ans =
        log(6)
 log(9) + pi*i
```

Other examples:

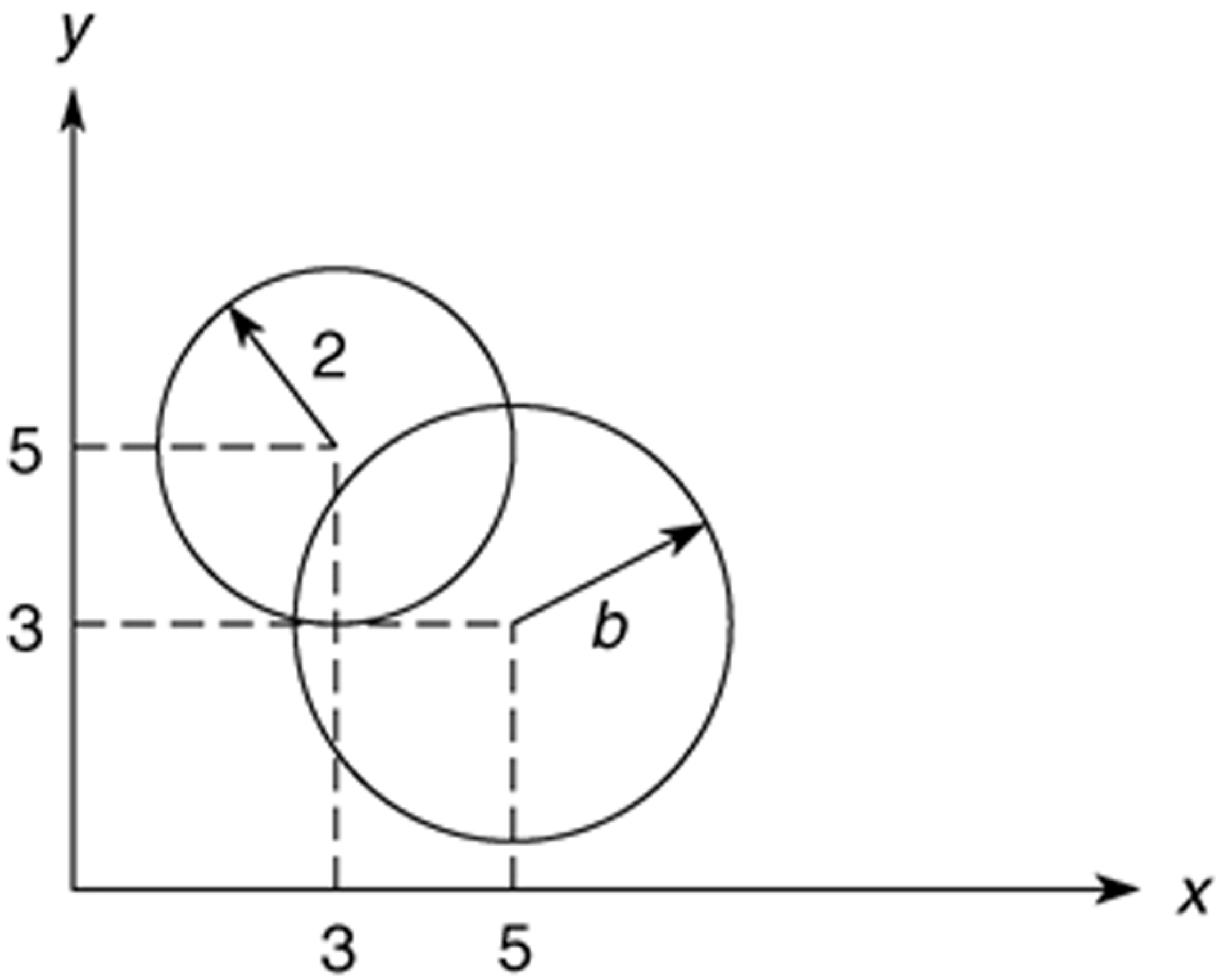
```
>> eq2 = 'y^2+3*y+2=0'; % quadratic eq
>> solve(eq2)
ans =
     [-2]
     [-1]

>> eq3 = 'x^2+9*y^4=0'; % x is squared
>> solve(eq3) % x is assumed the unknown
ans =
     [ 3*i*y^2]
     [-3*i*y^2]
```

When more than one variable occurs in the expression, MATLAB assumes that the variable closest to x in the alphabet is the variable to be found. You can specify the solution variable using the syntax

`solve(E, 'v')`, where `v` is the solution variable.

```
>> eq3 = 'x^2+9*y^4=0'; % y is to power 4
>> solve(eq3,'y')
ans =
   -((-1)^(1/4)*9^(3/4)*x^(1/2))/9
    ((-1)^(1/4)*9^(3/4)*x^(1/2))/9
  -((-1)^(1/4)*9^(3/4)*x^(1/2)*i)/9
   ((-1)^(1/4)*9^(3/4)*x^(1/2)*i)/9
```

Application of the `solve` function: Find the two Intersection points of the following two circles. Keep b unknown.



Solution

```
>> S = solve('(x-3)^2+(y-5)^2=4, (x-5)^2+(y-3)^2=b^2')
S =
    x: [2x1 sym]
    y: [2x1 sym]

>> S.x
ans =
 (- b^4/16 + (3*b^2)/2 - 1)^(1/2)/2 - b^2/8 + 9/2
 9/2 - b^2/8 - (- b^4/16 + (3*b^2)/2 - 1)^(1/2)/2

>> S.y
ans =
 (- b^4/16 + (3*b^2)/2 - 1)^(1/2)/2 + b^2/8 + 7/2
 b^2/8 - (- b^4/16 + (3*b^2)/2 - 1)^(1/2)/2 + 7/2
```

Differentiation with the `diff` function.

```
>> syms n x y
>> diff(x^n)
ans =
     x^n*n/x

>> simplify(ans)
ans =
     x^(n-1)*n

>> diff(log(x)) % means ln
ans =
     1/x

>> diff((sin(x))^2)
ans =
     2*sin(x)*cos(x)
```

If the expression contains more than one variable, the `diff` function operates on the variable *x*, or the variable closest to *x*, unless told to do otherwise. When there is more than one variable, the `diff` function computes the *partial* derivative.

```
>> syms x y
>> diff(sin(x*y))
ans =
     cos(x*y)*y
```

The function `diff(E,v)` returns the derivative of the expression `E` with respect to the variable `v`.

```
>> syms x y
>> diff(x*sin(x*y),y)
ans =
     x^2*cos(x*y)
```

The function `diff(E,n)` returns the *n*th derivative of the expression `E` with respect to the default independent variable.

```
>> syms x
>> diff(x^3,2)
ans =
     6*x
```

The function `diff(E,v,n)` returns the *n*th derivative of the expression `E` with respect to the variable `v`.

```
>> syms x y
>> diff(x*sin(x*y),y,2)
ans =
     -x^3*sin(x*y)
```

Integration with the `int` function.

```
>> syms x
>> int(2*x)
ans =
     x^2
```

The function `int(E)` returns the integral of the expression `E` with respect to the default independent variable.

```
>> syms n x y

>> int(x^n)
ans =
     x^(n+1)/(n+1)

>> int(1/x)
ans =
     log(x)

>> int(cos(x))
ans =
     sin(x)
```

$$\int x^n dx$$

$$\int \frac{1}{x} dx = \ln(x)$$

The form `int(E,v)` returns the integral of the expression `E` with respect to the variable `v`.

```
>>syms n x
>>int(x^n,n)
ans =
     1/log(x)*x^n
```

$$\int x^n dn$$

The form `int(E,a,b)` returns the integral of the expression `E` with respect to the default independent variable evaluated over the interval [*a*, *b*], where `a` and `b` are numeric expressions.

```
>>syms x
>>int(x^2,2,5)
ans =
     39
```

$$\int_{2}^{5} x^2 dx$$

The form `int(E,v,a,b)` returns the integral of the expression `E` with respect to the variable `v` evaluated over the interval [*a*, *b*], where `a` and `b` are numeric quantities.

```
>> syms x y
>> int(xy^2,y,0,5)
ans =
     125/3*x
```

The form `int(E,m,n)` returns the integral of the expression `E` with respect to the default independent variable evaluated over the interval [*m*, *n*], where `m` and `n` are symbolic expressions.

```
>> syms t x
>> int(x,1,t)
ans =
     t^2/2 - 1/2
```

$$\int_1^t x\,dx$$

```
>> syms t x
>> int(sin(x),t,exp(t))
ans =
     cos(t) - cos(exp(t))
```

The following session gives an example for which no integral can be found. The indefinite integral exists, but the definite integral does not exist if the limits of integration include the singularity at $x = 1$.

```
>> syms x
>> int(1/(x-1))
ans =
     log(x - 1)

>> syms x
>> int(1/(x-1),0,2)
ans =
    NaN
```

Taylor Series.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(t) + \frac{(x-a)^3}{3!}f^{(3)}(t) + \cdots$$

The `taylor(f,n,a)` function gives the first `n-1` terms in the Taylor series for the function defined in the expression `f`, evaluated at the point $x = a$. If the parameter `a` is omitted the function returns the series evaluated at $x = 0$.

```
>> syms x
>> f = exp(x);
>> taylor(f,3,2)
ans =
     exp(2)+exp(2)*(x-2)+(exp(2)*(x-2)^2)/2

>> taylor(f,4)
ans =
     x^3/6 + x^2/2 + x + 1
```

Series summation.

The `symsum(E,a,b)` function returns the sum of the expression `E` as the default symbolic variable varies from `a` to `b`.

```
>> syms k n
>> symsum(k,0,10)
ans =
     55
>> symsum(k^2, 1, 4)
ans =
     30
>> symsum(k,0,n-1)
ans =
     (n*(n - 1))/2
```

$$\sum_{k=0}^{10} k$$

$$\sum_{k=1}^{4} k^2$$

Finding limits.

The basic form `limit(E)` finds the limit as $x \rightarrow 0$.

```
>> syms a x
>> limit(sin(a*x)/x)
ans =
     a
```

The form `limit(E,v,a)` finds the limit as $v \to a$.

```
>>syms h x

>>limit((x-3)/(x^2-9),3)
ans =
     1/6

>>limit((sin(x+h)-sin(x))/h,h,0)
ans =
     cos(x)
```

The forms `limit(E,v,a,'right')` and `limit(E,v,a,'left')` specify the direction of the limit.

```
>> syms x
>> limit(1/x,x,0,'left')
ans =
     -inf

>> syms x
>> limit(1/x,x,0,'right')
ans =
     inf
```

Solving differential equations with `dsolve`

The `dsolve` syntax for solving a single equation is `dsolve('eqn')`. The function returns a symbolic solution of the ODE specified by the symbolic expression `eqn`.

```
>> dsolve('Dy+2*y=12')
ans =
     6+C1*exp(-2*t)
```

There can be symbolic constants in the equation.

```
>> dsolve('Dy=sin(a*t)')
ans =
     (-cos(a*t)+C1*a)/a
```

Here is a second-order example:

```
>> dsolve('D2y=c^2*y')
ans =
     C1*exp(-c*t) + C2*exp(c*t)
```

Sets of equations can be solved with `dsolve`. The appropriate syntax is `dsolve('eqn1','eqn2',...)`.

```
>>[x, y]=dsolve('Dx=3*x+4*y','Dy=-4*x+3*y')
   x =
C1*exp(3*t)*cos(4*t)+C2*exp(3*t)*sin(4*t)
   y = -
C1*exp(3*t)*sin(4*t)+C2*exp(3*t)*cos(4*t)
```

Conditions on the solutions at specified values of the independent variable can be handled as follows.

The form

```
dsolve('eqn', 'cond1', 'cond2',...)
```

returns a symbolic solution of the ODE specified by the symbolic expression `eqn`, subject to the conditions specified in the expressions `cond1`, `cond2`, and so on.

If `y` is the dependent variable, these conditions are specified as follows: `y(a) = b`, `Dy(a) = c`, `D2y(a) = d`, and so on.

Example:

```
>> dsolve('D2y=c^2*y','y(0)=1','Dy(0)=0')
ans =
     1/2*exp(c*t)+1/2*exp(-c*t)
```

Example:

```
>> [x,y]=dsolve('Dx=3*x+4*y','Dy=-4*x+3*y',
'x(0)=0','y(0)=1')

x =
  sin(4*t)*exp(3*t)
y =
  cos(4*t)*exp(3*t)
```

It is not necessary to specify only initial conditions. The conditions can be specified at different values of *t*.

```
>> dsolve('D2y+9*y=0','y(0)=1','Dy(pi)=2')
ans =
     cos(3*t) - (2*sin(3*t))/3
```

Laplace and Fourier Transform

```
>> syms b t
>> laplace(t^3)
ans =
     6/s^4

>> laplace(exp(-b*t))
ans =
     1/(s+b)

>> laplace(sin(b*t))
ans =
     b/(s^2+b^2)

>> fourier(exp(-t^2))
ans =
     pi^(1/2)/exp(w^2/4)
```

Laplace Inverse Transform

```
>>syms b s

>>ilaplace(1/s^4)
ans =
     1/6*t^3

>>ilaplace(1/(s+b))
ans =
     exp(-b*t)

>>ilaplace(b/(s^2+b^2)
ans =
     sin(b*t)
```

You can use the `inv(A)` and `det(A)` functions to invert and find the determinant of a matrix symbolically.

```
>> syms k
>> A = [0 ,1;-k, -2];
>> inv(A)
ans =
     [ -2/k, -1/k ]
     [ 1, 0 ]
>> A*ans     % verify inverse is correct
ans =
     [ 1, 0 ]
     [ 0, 1 ]
>> det(A)
ans =
     k
```

You can use matrix methods in MATLAB to solve linear algebraic equations symbolically. You can use the matrix inverse method, if the inverse exists, or the left-division method.

```
>> syms c
>> A = sym([2, -3; 5, c]);
>> b = sym([3; 19]);
>> x = inv(A)*b   % matrix inverse method
x =
 (3*c)/(2*c + 15) + 57/(2*c + 15)
 23/(2*c + 15)

>> x = A\b   % left-division method
x =
 (3*c)/(2*c + 15) + 57/(2*c + 15)
 23/(2*c + 15)
```

Homework

- Solve as many problems from Chapter 11 as you can
- Suggested problems:
- Solve: 11.3, 11.4, 11.12, 11.18, 11.22, 11.23, 11.28, 11.31, 11.32, 11.35, 11.37, 11.41, 11.42, 11.50, 11.51.